that the lens is diffraction limited, i.e., the exit pupil has been chosen so that the image of a point source is the Airy intensity distribution.

But first, we remark that the intensity in this problem is related to that of the complementary case, the image of an opaque disc of radius $a$. According to Babinet’s principle, the sum of the light amplitudes for these two cases is the constant light amplitude without either. (This is easy to see from the linearity of the Huygens-Fresnel-Fraunhofer expression discussed in Appendix B3). So, where one is light, the other is dark.

Suppose the hole is illuminated with incoherent light, as in ordinary microscopy. If $a < \lambda/4$, the illumination is nonetheless effectively coherent, since any incident plane wave of random phase will have little phase difference across the hole. If $a > \lambda/2$, the illumination may be considered incoherent. This is the case considered here.

1. Incoherent Illumination

If this were geometrical optics, light from each uniformly illuminated point of the object plane would pass through the lens and be focused as an illuminated point on the image plane. The properly scaled image of all these illuminated points would be a uniformly illuminated circle of radius $a$. We shall call the circumference of this circle the “image circle edge.” The new wrinkle is that diffraction surrounds each imaged point with its own Airy disc (assuming that spherical aberration is negligible), so that the image extends beyond the image circle edge. The intensities add so, at a point $r$ on the image plane, the net intensity is

$$I(r) \sim \frac{1}{\pi} \int_{A_0} \frac{J_1(k\tilde{b}|r - r_0|)}{|r - r_0|}^2, \quad (G1)$$

where $A_0$ is the area of the image circle, and $\tilde{b} \equiv b/f$ is called the numerical aperture.

For $a >> r_A$, where $r_A$ is the Airy radius, the intensity at the center point of the image circle is, by (G1),

$$I(0) \sim \frac{1}{\pi} \int_0^a r_0 dr_0 \int_0^{2\pi} \, d\phi \left[ \frac{J_1(k\tilde{b}r_0)}{r_0} \right]^2 \approx 1.$$  

In this equation, the limit $a$ has been extended to $\infty$ with no appreciable error, since the major contribution is from Airy discs centered within distance $r_A$ of the origin.

As the point of interest moves off center, the intensity remains essentially constant, until at a distance $\approx a - r_A$ from the center, a distance $r_A$ from the image circle edge. Then $I$ starts to decrease, reaching the value $\approx .5$ at the edge. This is because, at the edge, $\approx$half the Airy discs contribute intensity, compared to the discs which contribute intensity at a point well inside the image circle.

Now, we turn to quantitative analysis of the general case, with no restriction of the relative sizes of $a$ and $r_A$. We shall calculate the intensity (G1) outside the image circle, at $r = 0$ which is placed a distance $z$ beyond the image circle edge, i.e., the center of the image circle in this coordinate system is at $r = a + z$. The contributing Airy disc centers lie within the image circle, between radius $r_0 (z \leq r_0 \leq 2a + z)$ and radius $r_0 + dr_0$, along an arc subtending an angle $2\phi$. The hole circumference $(x-a)^2 + y^2 = a^2$ cuts this arc at two points. Setting $x = r_0 \cos \phi$ and $y = r_0 \sin \phi$ in this expression allows one to find $\cos \phi$. Eq. (G1) becomes

$$I_{\text{out}}(z) \sim \frac{2}{\pi} \int_z^{2a+z} r_0 \cos^{-1} \left[ \frac{r_0^2 + z^2 + 2az}{2r_0(a + z)} \right] J_1^2(k\tilde{b}r_0) r_0 \cos \phi, \quad (G2)$$

For completeness, we put here the comparable expression for the intensity inside the image circle. Again, we calculate the intensity (G1) at $r = 0$, where this new coordinate system origin is a distance $z$ away from the center of the image circle. There are two contributions, one from a circular area of radius $a - z$, the other from the rest of the disc ($a - z \leq r_0 \leq a + z$):

$$I_{\text{in}}(z) \sim 2 \int_0^{a-z} r_0 dr_0 J_1^2(k\tilde{b}r_0) r_0 \cos^{-1} \left[ \frac{r_0^2 + z^2 - a^2}{2r_0 z} \right] J_1^2(k\tilde{b}r_0) r_0, \quad (G3)$$

For large $a$, (G2) becomes

$$I_{\text{out}}(z) \approx \frac{2}{\pi} \int_z^{\infty} r_0 \cos^{-1} \left[ \frac{z}{r_0} \right] J_1^2(k\tilde{b}r_0) r_0.$$

This is a function of $k\tilde{b}z = 3.83(z/r_A)$. Numerical evaluation shows $I_{\text{out}}(z)$ drops from $.5$ at $z = 0$ to $\approx .05$ at $z = r_A$. While it is somewhat subjective, this suggests that we take the perceived edge of the image of the hole to be located where the intensity is 5% of its maximum value at the center of the image circle. Thus, diffraction increases the radius of a large hole from $a$ to $R \approx a + r_A$.

By changing the variable of integration in (G2) to $r_0/a$, one sees that the intensity is a function of two variables, $z/a$ and $kba/3.83 = a/r_A$. For each value of $a/r_A$, one can numerically solve Eq. (G2) for the value of $z/a$ for which $I(z) = .05I(0)$. This is the ratio $R/a$, where $R$ is defined as the radius of the image. A graph of $R/r_A$ vs $a/r_A$ is given in Fig. 12, and is discussed in section III H.


[2] A transcript of an interesting historical talk about
the Lewis and Clark expedition and its genesis, by Robert S. Cox, given at Monticello in 2004, may be found at http://www.monticello.org/streaming/speakers/transcripts/cox.html


[4] A wealth of material on the Lewis and Clark expedition and related topics appears at http://www.lewis-clark.org/. For a discussion of Clarkia pulchella, a picture of the plant, and Pursh’s drawing of it which appeared in his Flora, click on “Natural History,” then “Plants,” then “Clarkia.” It is also interesting to read here about the lives (and tragic deaths of the first three) of Lewis (see also http://www.prairiehosts.com/meriwet.html), Pursh (see also J. Ewan, Proc. Amer. Phil. Soc., 96 #5, 599 (1952)), available at Googlebooks), Douglas (see also http://www.coffeetimes.com/daviddouglas.htm) and Barton, all by the American botanist J. Reveal.


[10] We are indebted to Bronwen Quarry of the Hudson’s Bay Company archives for furnishing information about the dates of travel and fate of the William and Ann.
[15] http://www.microscopy-uk.org.uk/dww/home/hom-brown.htm also has a video of milk fat droplets undergoing Brownian motion, of sizes .5 to 3μm, with tips on how to duplicate the observation. It incorrectly states: “...he noticed the motion in tiny particles suspended within the medium of living pollen grains.” although correctly adding “Some textbooks, even to university level, incorrectly state that he observed the movement of the pollen grains themselves. Most pollen grains are too large to exhibit noticeable Brownian motion.”
[17] A biography of John Lindley is available at the orchid information web site http://www.orchids.co.in/: select orchidolologists.
[18] The Horticultural Society garden in Chiswick was part of the 655 acre estate of the lessee, William Spencer Cavendish (1790-1858), the 6th Duke of Devonshire, with a gate to the garden installed for the Duke’s use. The physicist Henry Cavendish (1731-1810) was related, a grandson of the 2nd Duke. What remains of this estate, Chiswick House on 68 acres, a fine example of Palladian architecture, is being restored and can be visited (http://www.chgt.org.uk). More of the acreage exists as a Chiswick public park, Dukes Meadows (http://www.dukesmeadowtrust.org/). However, the Horticultural Society garden is no more, replaced by city streets.
[19] We are indebted to David Mabberley for “strongly” suggesting that the slips catalog be looked into, and to Armando Mendez, of the Botany Library of the National History Museum, for finding this sheet and mailing us a copy.
[20] We are indebted to Brent Elliot, Historian of the Royal Horticultural Society, for this information. The society does not have any surviving plant receipt books for the garden at Chiswick until the 1840s, so one cannot track Clarkia pulchella into or out of the garden. Lindley was in overall supervision of the gardens, with Donald Munro as head gardener, and a staff of under-gardeners. It was common for botanists to share plants of interest.
[21] The complete entries for both dates are reproduced in J. Ramsbottom, Journ. of Bot, 70 13, (1932).
[27] This is a translation from the French. The original appears in Mabberley, Ibid, pp. 157-171.
lens,” Microscope 138 (April, 2007): this is an online article available at http://www.microscopy-uk.org.uk. Select the Library/Issue Archive.


[34] B. J.Ford, The Microscope 40, 235 (1992) verified that Brown could have seen motion with his Linnean x170 lens, writing: “The Clarkia pollen was obtained from author's of C. pulchella at the Botanical Garden at Cambridge University, and pollen specimens from other species within the Oenotheraceae were also utilized. Exactly as Brown recorded, the experiments were carried out in the month of June and the pollen grains were mounted in water after removal from pre-deshinet anthers. ... The phenomenon of Brownian movement was well resolved by the original microscope lens. Within the pollen grains, ceaseless movement could be observed.” However, while the author saw the contents of the pollen, he did not see them undergoing Brownian motion, as this appears to suggest. Rather, with this lens, he observed the Brownian motion of similarly sized milk fat droplets (private communication from B. J. Ford). Presumably, what was meant is that “ceaseless movement could have been observed” with that lens. Perhaps this led to the erroneous statement in an article reporting on this work, Science News 142 (Aug. 15), 109 (1992): “Each grain contains a thick liquid ... When Brown looked inside the pollen grains with his microscope, he could see tiny particles, each about 1 micron across, suspended in the liquid and constantly in motion.”


[56] D. E. Bilderback, Ibid.


[61] Here are some web sites that describe how to observe pollen tubes in the laboratory: http://www-saps.plantsci.cam.ac.uk/pollen/pollen2.htm,
http://www.saps.plantsci.cam.ac.uk/
worksheets/ssheets/sssheet4.htm, and
http://www.microscopy-uk.org.uk/mag/artdec99
/jgpollen.html.


[63] I grew Clarkia plants from Diane’s seeds, http://www.dianeseeds.com: packets of C. pulchella (containing ≈1000 seeds), C. amoena (≈1800 seeds) and C. elegans (≈1500 seeds) each cost $2.00. Everwilde farms, http://www.everwilde.com sells packets (of ≈2000 seeds) of the latter two species for $2.50. Thompson and Morgan, http://www. tmsseeds.com, a British company, sells packets of C. pulchella and C. elegans (≈400 seeds) for $2.55. Monticello sells a packet of seeds of C. pulchella (a plant cultivated by Thomas Jefferson) for $2.50, but I did not have good results with these seeds.

[64] The U. S. Department of Agriculture site, http://plants.usda.gov/, has much general information on species. Under Scientific Name, type in Clarkia, and then choose Clarkia Pursh.


[66] Hydrofarm Green Thumb or Jump Start (they seem to be the same) fixtures with bulbs are available from various vendors. For example, DirtWorks, http://www.dirtworks.net/Grow-Lights.html, sells the fixture, the 2 foot version (which will light one Lee Valley Seed Starter) costs $60 plus shipping and the 4-foot version (which I bought and which lights three Lee Valley Seed Starters) costs $89 plus shipping.

[67] Image is available from the National Institutes of Health web site http://rsweb.nih.gov/ij/.

[68] P. W. van der Pas, Scientarium Historia 13, 127 (1971): Of Brown’s molecules, van der Pas says: “... they were approximately of the same size; their diameter varying between 1.26 and 1.6 microns. These statements are not true, BROWN was led to them because he worked with an imperfect lens at the border line of its magnifying power.” This seems to be the only place to find this latter assertion which, however, van der Pas did not enlarge upon. He wrote this paper to call attention to a rather throw-away paragraph in a paper in 1784 by Jan Ingenhousz, thereby intimating Ingenhousz’s priority over Brown. The purpose of Ingenhousz’s paper was to introduce the idea of a transparent cover slip in microscopy to prevent water evaporation. Unlike others who had seen Brownian motion before Brown, but attributed it to life, Ingenhousz observed and clearly asserted in this paragraph that nonliving matter underwent the motion, but he did no systematic investigation.

Citing van der Pas, Mabberley (Op. Cit. p. 272) says: “It has been shown that Brown’s ‘molecules’ were artifacts, there being no particles 1.26-1.6 μm across in pollen grains or elsewhere.” However, this statement is only partially correct. While there are not *universal* particles of this size range as Brown supposed, spherosomes imaged to such size were certainly seen by Brown. Spherosomes have been observed in various plant tissues of diameter .4-1 μm[69].


[72] The polystyrene latex spheres were obtained from Ted Pella, Inc., http://www.tedpella.com/.


[74] Edmund Optical Company, 1 mm diameter ball lens #NT43-708, costing $22.


[83] A simpler problem is the scattering of a plane wave by a sphere. The solution for our problem (scattering of a wave emerging from a point source) is a superposition of solutions of the plane wave problem. However, this “simpler” problem is itself quite complicated. Its solution can be written exactly, as an infinite sum of angular momentum eigenstates, each with a spherical Bessel function giving the radial behavior. Since λ <<< R, one cannot truncate the series at a few terms. Sophisticated techniques (such as the Watson transform, Regge pole theory, method of steepest descent) are used to sum appropriate terms corresponding to the physical behavior of rays, first discussed by Debye. The first sum corresponds to reflection from the sphere, the second to the geometrical optics refraction and its attendant aberrations, the third to one internal reflection (responsible for the behavior of the rainbow), the fourth to two internal reflections (responsible for the behavior of the glory), etc: H. M. Nussenzveig, Journ. Math. Phys. 10, 82 & 125 (1969). The sum for the electromagnetic field is called the Mie solution. Mie solution calculators, which sum the terms
numerically, are available on the web. For an analytic treatment, see W. T. Grandy Jr., *Scattering of waves from large spheres* (Cambridge U. P., Cambridge 2005).
